

Self-similar factor approximants for evolution equations and boundary-value problems

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Abstract

The method of self-similar factor approximants is shown to be very convenient for solving different evolution equations and boundary-value problems typical of physical applications. The method is general and simple, being a straightforward two-step procedure. First, the solution to an equation is represented as an asymptotic series in powers of a variable. Second, the series are summed by means of the self-similar factor approximants. The obtained expressions provide highly accurate approximate solutions to the considered equations. In some cases, it is even possible to reconstruct *exact* solutions for the whole region of variables, starting from asymptotic series for small variables. This can become possible even when the solution is a transcendental function. The method is shown to be more simple and accurate than different variants of perturbation theory with respect to small parameters, being applicable even when these parameters are large. The generality and accuracy of the method are illustrated by a number of evolution equations as well as boundary value problems.

PACS numbers: 02.30.Hq, 02.30.Mv, 02.60.Lj, 03.75.Lm, 05.45.Yv

Keywords: Evolution equations; Boundary-value problems; Solitons; Bose-Einstein condensate; Vortices

1 Introduction

Differential equations appear in numerous problems of physics, applied mathematics [1–3] and many other branches of natural as well as social sciences (see, e.g., [4–7]). In the majority of cases, these equations are nonlinear and cannot be solved exactly, allowing for exact solutions only in a few exceptional instances. Then, in order to obtain approximate analytical solutions, one resorts to perturbation theory in powers of some parameters assumed to be small [1–3]. The resulting expressions are usually rather cumbersome and are difficult to analyze. Also, they form asymptotic series that are useful only for very small expansion parameters.

The validity of perturbative series can be extended to the finite values of parameters by reorganizing them with the help of the optimized perturbation theory [8]. The basic idea of this theory is to include in the initial approximation a set of auxiliary parameters which are transformed, at each step of perturbation theory, into control functions governing the series convergence [8]. The optimized perturbation theory has been successfully applied to a great variety of problems, providing rather accurate approximations (see review article [9] and references therein). It has also been applied to solving differential equations [10–17]. However, the weak point of this approach to finding the solutions of differential equations is that the optimization procedure results in very complicated equations for control functions, which are to be solved numerically. It is practically always much easier to solve the given differential equations numerically than to deal with the cumbersome optimization equations for control functions, anyway requiring numerical solution. This is why this approach, though being very useful for many other problems [9], has not found wide practical use for solving differential equations.

Another method of constructing approximate solutions is based on the self-similar approximation theory [18–26]. Then the solutions to differential equations can be represented in the form of self-similar exponential approximants or self-similar root approximants [27–34]. These approximants represent well those functions whose behavior at large variables is known to be either exponential or power-law, respectively.

In the present paper, we advocate a novel approach to constructing approximate solutions of differential equations. This approach is based on the use of *self-similar factor approximants* [35–37]. The mathematical derivation of the latter also rests on the self-similar approximation theory [18–26], but the structure of these approximants is rather different from the exponential and root approximants [27–34]. The structure of the self-similar factor approximants reminds that of thermodynamic characteristics near critical points [38,39]. This is why it has been natural to apply, first, these approximants to the description of critical phenomena [35–37,40]. It was shown that these approximants allow us a straightforward and simple determination of critical points and critical indices, agreeing well with the results of the most complicated numerical techniques, whose description can be found in articles [41–46] and books [47–49].

The self-similar factor approximants make it possible to define an effective sum of divergent series. Initially, these approximants were introduced [35–37] for summing the partial series of even orders, while the summation of odd-order series was not defined. Recently, the method was completed by defining the factor approximants of odd orders [40,50]. Now we have in hands a general and uniquely prescribed procedure for constructing the self-similar

factor approximants of arbitrary orders. We show below that this procedure can be employed for finding very accurate approximate solutions to differential equations. In some cases, the method gives exact solutions, if these exist. The approach is very general, being applicable to linear as well as to nonlinear equations, and to initial-value as well as to the boundary-value problems. The principal difference of the method from other perturbation theories [1–3] is that we, first, represent the sought solutions as an asymptotic series in powers of the equation variable, but not in powers of a parameter. And, second, we extrapolate the given series to the whole range of the variable by means of the self-similar factor approximants. The advantage of the method is its extreme simplicity combined with the high accuracy of the obtained solutions.

2 Self-similar factor approximants

Here we describe the general procedure of constructing self-similar factor approximants as solutions to differential equations. We keep in mind ordinary differential equations, though the procedure can be generalized to partial differential equations. It would be unreasonable to plunge from the very beginning to complicated matters, but the principal idea should, first, be illustrated by not too complex problems. In line with this, we consider the case, where the solution is a real function of a real variable. Thus, we assume that we are interested in finding a solution to a differential equation, which is a real function $f(x)$ of a real variable x . First, we are looking for the solution in the region of asymptotically small $x \rightarrow 0$, where it can be represented as an asymptotic series

$$f(x) \simeq f_0(x) \sum_n a_n x^n \quad (x \rightarrow 0) , \quad (1)$$

where $f_0(x)$ is a given function. It is worth emphasizing that the expansion variable x is not compulsory the given independent variable, but can be a function of it, which means that one can always accomplish the change of variables and look for the expansion in powers of the new variable. For instance, if the initial variable entering an equation is t , we are not obliged to look for an expansion necessarily in powers of t , but we can make a change of the variable, introducing $x = x(t)$ and to study an expansion in powers of $x(t)$. In particular, $x(t)$ could be t^α . If α is not an integer, we get the Puiseux expansion. In general, one can consider asymptotic expansions of the form $\sum_n a_n x_n$, with functions x_n from the given comparison scale [51].

Suppose, series (1) is limited by a finite sum of k -th order

$$f_k(x) = f_0(x) \sum_{n=0}^k a_n x^n , \quad (2)$$

with $k = 0, 1, 2, \dots$. Without the loss of generality, we may set $a_0 = 1$, since if a_0 would not be 1, we could include it in $f_0(x)$.

The self-similar factor approximants of even orders $k = 2p$, with $p = 1, 2, \dots$, are defined [35–37] as

$$f_{2p}^*(x) = f_0(x) \prod_{i=1}^p (1 + A_i x)^{n_i} . \quad (3)$$

The parameters A_i and n_i are obtained from expanding Eq. (3) in powers of x and comparing the resulting expansion with sum (2) by the accuracy-through-order procedure. The corresponding equations for a_i and n_i can be written as

$$\sum_{i=1}^p n_i A_i^n = B_n \quad (n = 1, 2, \dots, 2p) , \quad (4)$$

where

$$B_n = \frac{(-1)^{n-1}}{(n-1)!} \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \ln \sum_{m=0}^k a_m x^m . \quad (5)$$

Equations (4), whose number is $2p$, define all $2p$ unknowns A_i and n_i . The factor approximants (3) extrapolate expansion (2), valid for small $x \rightarrow 0$, to the region of finite x .

For odd orders $k = 2p + 1$, with $p = 0, 1, 2, \dots$, the factor approximants are defined [40,50] in the form

$$f_{2p+1}^*(x) = f_0(x) \prod_{i=1}^{p+1} (1 + A_i x)^{n_i} , \quad (6)$$

in which $A_1 = 1$ and all other $2p + 1$ unknown parameters are given by $2p + 1$ equations

$$\sum_{i=1}^{p+1} n_i A_i^n = B_n \quad (n = 1, 2, \dots, 2p + 1) , \quad (7)$$

with B_n from Eq. (5).

The index k , labelling the factor approximants $f_k^*(x)$, is determined by the order k of $f_k(x)$ in sum (2), which is used for calculating the parameters A_i and n_i . It may happen that there exist additional conditions imposed on the behavior of the sought function $f(x)$. Then the structure of $f_k^*(x)$ should include additional factors, whose parameters are chosen so that to satisfy the imposed conditions. In the case of differential equations, such additional restrictions are given by initial and boundary conditions.

For example, the Cauchy initial value problems are supplemented by the set of initial conditions for the sought function and its derivatives

$$f(0) = f_0 , \quad \left[\frac{d^m f(x)}{dx^m} \right]_{x=0} = f_0^{(m)} , \quad (8)$$

where $m = 1, 2, \dots$. Therefore, the factor approximants are constructed so that to satisfy the same initial conditions

$$f_k^*(0) = f_0 , \quad \left[\frac{d^m f_k^*(x)}{dx^m} \right]_{x=0} = f_0^{(m)} \quad (9)$$

for all orders of k .

In the case of a boundary value problem, for a function $f(x)$ on an interval of $x \in [x_1, x_2]$, one has the boundary conditions

$$f(x_1) = f_1 , \quad f(x_2) = f_2 . \quad (10)$$

Respectively, the factor approximants of each order have to satisfy the same boundary conditions

$$f_k^*(x_1) = f_1, \quad f_k^*(x_2) = f_2. \quad (11)$$

When a boundary value problem is formulated for an infinite interval, the related boundary conditions are given in the form of the asymptotic behavior of the function $f(x)$. For instance, let the asymptotic form of $f(x)$ at large x be prescribed as

$$f(x) \simeq Bx^\beta \quad (x \rightarrow \infty), \quad (12)$$

where $B \neq 0$ and β is any real quantity, including zero. Hence, the factor approximants $f_k^*(x)$ of any order must possess the same asymptotic form

$$f_k^*(x) \simeq Bx^\beta \quad (x \rightarrow \infty). \quad (13)$$

Assume that the zero-order factor in Eqs. (1) and (2) behaves at large x as

$$f_0(x) \simeq cx^\alpha \quad (x \rightarrow \infty), \quad (14)$$

with $c \neq 0$ and any real α . Then, in order that the factor approximants (3) and (6) would satisfy the asymptotic boundary condition (13), it should be that

$$c \prod_i A_i^{n_i} = B, \quad \alpha + \sum_i n_i = \beta. \quad (15)$$

Depending on whether initial-value or boundary-value problems are considered, we shall always require that the corresponding conditions be valid *exactly*. This distinguishes our approach from perturbation theories with respect to some parameters, when boundary conditions are usually satisfied only approximately [1–3].

3 Initial value problems

Considering the Cauchy initial value problems, we shall concentrate our attention on the so-called singular problems, where a parameter enters in front of the highest derivative [1–3]. When one tries to apply perturbation theory with respect to the given parameter, one confronts nonuniformly valid expansions [1–3]. However in our approach, we start with an expansion not in powers of this parameter, but in powers of the equation variable. Then, using the extrapolation by self-similar factor approximants, we obtain accurate solutions for any value of the parameter. Moreover, in some cases, the factor approximants restore *exact* solutions. This fact is so unusual that we illustrate it in detail below.

A. Linear singular problem

Let us consider the singular initial value problem

$$\varepsilon \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + \frac{y}{\varepsilon} = 0, \quad (16)$$

with the initial conditions

$$y(0) = 1 , \quad \left(\frac{dy}{dt} \right)_{t=0} = - \frac{1}{\varepsilon} , \quad (17)$$

defining $y = y(t)$ for $t \geq 0$. And let us pretend that we do not know its exact solution.

Following our approach, we, first, look for the behavior of the solution $y(t)$ at asymptotically small $t \rightarrow 0$, which is

$$y(t) \simeq 1 - \frac{t}{\varepsilon} + \frac{t^2}{2\varepsilon^2} + \dots \quad (t \rightarrow 0) . \quad (18)$$

For the factor approximant of second order, we get

$$y_2^*(t) = \lim_{A \rightarrow 0} (1 + At)^{-1/A\varepsilon} = e^{-t/\varepsilon} , \quad (19)$$

which is the exact solution of Eq. (16). Similarly, for the factor approximants of higher orders after $k = 3, 4, 5$ and so on, we find

$$y_k^*(t) = e^{-t/\varepsilon} \quad (k \geq 3) , \quad (20)$$

that is, the *exact* solution of Eq. (16) under the initial conditions (17) and any real ε .

It is easy to notice that function (20) cannot in principle be expanded in terms of ε . The functions of this type are called instanton solutions [47,52]. Note also that the exponential (20) is a transcendental function.

B. Nonlinear singular problem

The model problem of carrier transfer [2,3] is given by the nonlinear equation

$$(\varepsilon y + t) \frac{dy}{dt} + y - 1 = 0 \quad (21)$$

for $y = y(t)$ and $t \geq 0$, with the initial condition

$$y(0) = 2 . \quad (22)$$

We again pretend that we are not aware of the solution to Eq. (21). If we try to invoke perturbation theory with respect to the parameter ε , we come to a rather nontrivial singular problem. The method of strained coordinates or the method of asymptotic matching could be used, but with a necessity of dealing with very lengthy and cumbersome calculations [2,3]. We show here that this problem is easily solvable by our method.

As is explained in Sec. 2, it is always possible to make a change of variables, which seems to be convenient. If we introduce the function $z(x)$, such that

$$z = x + y , \quad x = \frac{t}{\varepsilon} , \quad (23)$$

then Eq. (21) simplifies to

$$z \frac{dz}{dx} - x - 1 = 0 , \quad (24)$$

with the initial condition

$$z(0) = 2 . \quad (25)$$

The asymptotic behavior of $z(x)$ at small x is

$$z(x) \simeq 2 + \frac{1}{2} x + \frac{3}{16} x^2 + \dots \quad (x \rightarrow 0) . \quad (26)$$

Constructing the factor approximants, we find that, starting from the fourth order, the approximants $z_k^*(x)$ provide the exact solutions to Eq. (24), with the initial condition (25). Thus, in fourth order, we have

$$z_4^*(x) = 2(1 + A_1 x)^{n_1} (1 + A_2 x)^{n_2} . \quad (27)$$

From the accuracy-through-order procedure, with the given initial condition (35), we get

$$A_1 = \frac{1}{4} (1 - i\sqrt{3}) , \quad A_2 = A_1^* , \quad n_1 = n_2 = \frac{1}{2} . \quad (28)$$

Thence Eq. (27) reduces to

$$z_4^*(x) = \sqrt{4 + 2x + x^2} , \quad (29)$$

which is the exact solution of Eq. (24). Acting in the same way for higher-order approximants, and using relations (23), we come to the factor approximants

$$y_k^*(t) = \sqrt{4 + \frac{2t}{\varepsilon} + \frac{t^2}{\varepsilon^2}} - \frac{t}{\varepsilon} \quad (k \geq 4) , \quad (30)$$

giving the *exact* solution of Eq. (21) for orders $k \geq 4$, any real ε , and all $t \geq 0$.

C. Singular logistic equation

The logistic equation and its variants are widely employed in the studies of population dynamics [53]. For the population $p(t)$, being a function of time t , with the growth rate $1/\varepsilon$, the logistic equation writes as

$$\varepsilon \frac{dp}{dt} = p(1 - p) . \quad (31)$$

It is given for $t \geq 0$, with the initial condition

$$p(0) = p_0 . \quad (32)$$

If again we pretend not to know the solution of Eq. (31) and try to resort to perturbation theory in powers of the large growth rate, that is, small ε , we confront a singular value problem.

Wishing to apply our approach, it is reasonable, as has been discussed above, to choose the most convenient variables. The hint on what variables would be the most convenient comes from analyzing the dynamics in the vicinity of the stationary points. Equation (31) possesses one stable fixed point $p(\infty) = 1$. The motion near this point is given by the asymptotic law

$$p(t) \simeq 1 + ce^{-t/\varepsilon} \quad (t \rightarrow \infty) . \quad (33)$$

The latter suggests to choose as the variable

$$x \equiv e^{-t/\varepsilon} . \quad (34)$$

Considering the population $p(t)$ as a function of this variable (34), for the function

$$y = p(t(x)) = y(x) , \quad (35)$$

we have the equation

$$\frac{dy}{dx} = y(y-1)x , \quad (36)$$

with the initial condition

$$y(1) = p_0 . \quad (37)$$

The small x -expansion of $y(x)$ is

$$y(x) \simeq \sum_n a_n x^n \quad (x \rightarrow 0) , \quad (38)$$

for which Eq. (36) yields $a_n = a_1^n$. The lowest-order factor approximant, for which the initial condition can be satisfied, is the third-order approximant resulting in the expression

$$y_3^*(x) = (1 + A_1 x)^{n_1} , \quad (39)$$

where

$$A_1 = \frac{1}{p_0} - 1 , \quad n_1 = -1 .$$

The same expression (39) follows for other approximants of higher orders $k \geq 3$. Returning to the population function $p(t)$ by means of relations (34) and (35), we find

$$p_k^*(t) = \frac{p_0}{p_0 - (p_0 - 1)e^{-t/\varepsilon}} \quad (k \geq 3) . \quad (40)$$

This is the *exact* solution of the logistic equation (31), with the initial condition (32).

4 Boundary value problems

In the previous section, we have shown that the method of self-similar factor approximants allows us to reconstruct exact solutions of some differential equations representing initial value problems. This highly nontrivial fact occurs as well for some nonlinear boundary value problems.

A. Kink soliton equation

Nonlinear equations possessing soliton solutions are met in various problems of physics and applied mathematics. Let us consider, for example, the nonlinear Schrödinger equation describing the so-called φ^4 -model with particle mass $1/\varepsilon$. The equation reads as

$$\frac{\varepsilon}{2} \frac{d^2 \varphi}{dx^2} + \varphi - \varphi^3 = 0 . \quad (41)$$

Assume that the function $\varphi = \varphi(x)$ satisfies the boundary conditions

$$\varphi(-\infty) = -1, \quad \varphi(\infty) = 1. \quad (42)$$

From Eq. (41) and the boundary conditions (42) it follows that $\varphi(x)$ is an antisymmetric function, such that

$$\varphi(-x) = -\varphi(x), \quad (43)$$

and, therefore,

$$\varphi(0) = 0. \quad (44)$$

To choose a convenient expansion variable, we again study the form of the function $\varphi(x)$ in the vicinity of the stable stationary points $\varphi(\mp\infty) = \mp 1$, where we have

$$\begin{aligned} \varphi(x) &\simeq -1 + a \exp\left(\frac{2}{\sqrt{\varepsilon}} x\right) & (x \rightarrow -\infty), \\ \varphi(x) &\simeq 1 - a \exp\left(-\frac{2}{\sqrt{\varepsilon}} x\right) & (x \rightarrow \infty), \end{aligned} \quad (45)$$

with a being a real parameter. This immediately suggests to choose as a variable

$$z \equiv \exp\left(\frac{2}{\sqrt{\varepsilon}} x\right). \quad (46)$$

In terms of the latter variable, Eqs. (45) become

$$\begin{aligned} \varphi(x) &\simeq -1 + az & (z \rightarrow 0, \ x \rightarrow -\infty), \\ \varphi(x) &\simeq 1 - \frac{a}{z} & (z \rightarrow \infty, \ x \rightarrow 0). \end{aligned} \quad (47)$$

It is also convenient to introduce the positively defined function

$$y \equiv 2 + \varphi = y(z), \quad (48)$$

satisfying the boundary conditions

$$y(0) = 1, \quad y(1) = 2. \quad (49)$$

Then Eq. (41) transforms into

$$2z^2 \frac{d^2 y}{dz^2} + 2z \frac{dy}{dz} + 6 - 11y + 6y^2 - y^3 = 0. \quad (50)$$

Looking for the form of $y(z)$ at small $z \rightarrow 0$, we substitute the asymptotic expansion

$$y(z) \simeq \sum_n a_n z^n \quad (z \rightarrow 0) \quad (51)$$

into Eq. (50), from where it follows that

$$a_0 = 1, \quad a_n = \frac{(-1)^{n-1}}{2^{n-1}} a_1^n \quad (n \geq 1). \quad (52)$$

Constructing the factor approximants from expansion (51), we obtain the same expression

$$y_k^*(z) = \frac{2 + 3a_1 z}{2 + a_1 z} \quad (k \geq 4) \quad (53)$$

for orders $k \geq 4$, with the constant $a_1 = 2$ found from the boundary conditions (49). Using relations (46) and (48), according to which

$$\varphi(x) = y(z(x)) - 2 , \quad (54)$$

we come to the expression

$$\varphi_k^*(x) = \tanh\left(\frac{x}{\sqrt{\varepsilon}}\right) \quad (k \geq 4) . \quad (55)$$

This is the *exact* kink solution of the soliton Eq. (41).

Recall that, if we would try to solve Eq. (41) by means of perturbation theory with respect to large mass, that is, small ε , we would have to deal with a rather unpleasant and cumbersome boundary-layer problem [1–3].

B. Bell soliton equation

The nonlinear Schrödinger equation for a negative mass $-1/\varepsilon$ is

$$\frac{\varepsilon}{2} \frac{d^2 \varphi}{dx^2} - \varphi + \varphi^3 = 0 . \quad (56)$$

The function $\varphi = \varphi(x)$ satisfies the boundary conditions

$$\varphi(-\infty) = 0 , \quad \varphi(\infty) = 0 . \quad (57)$$

From these it follows that $\varphi(x)$ is a symmetric function,

$$\varphi(-x) = \varphi(x) , \quad (58)$$

such that

$$\lim_{x \rightarrow \pm\infty} \frac{d\varphi}{dx} = 0 . \quad (59)$$

In the vicinity of the stationary points $\varphi(\mp\infty) = 0$ we have

$$\begin{aligned} \varphi(x) &\simeq a \exp\left(\sqrt{\frac{2}{\varepsilon}} x\right) \quad (x \rightarrow -\infty) , \\ \varphi(x) &\simeq a \exp\left(-\sqrt{\frac{2}{\varepsilon}} x\right) \quad (x \rightarrow \infty) . \end{aligned} \quad (60)$$

Hence the appropriate convenient variable here is

$$z \equiv \exp\left(\sqrt{\frac{2}{\varepsilon}} x\right) . \quad (61)$$

In terms of the latter, Eq. (60) simplify to

$$\begin{aligned}\varphi(x) &\simeq az & (z \rightarrow 0, \ x \rightarrow -\infty) , \\ \varphi(x) &\simeq \frac{a}{z} & (z \rightarrow \infty, \ x \rightarrow \infty) .\end{aligned}\tag{62}$$

For the function

$$y = \varphi(x(z)) = y(z)\tag{63}$$

of variable (61), from Eq. (56), one gets the equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} - y + y^3 = 0 ,\tag{64}$$

with the boundary conditions

$$y(0) = 0 , \quad y(\infty) = 0 .\tag{65}$$

And property (59) takes the form

$$\lim_{z \rightarrow 1} \frac{dy}{dz} = 0 .\tag{66}$$

In the asymptotic region of small z , the solution to Eq. (64) is

$$y(z) \simeq \sum_n a_n z^n \quad (z \rightarrow 0) ,\tag{67}$$

in which

$$\begin{aligned}a_{2n} &= 0 & (n = 0, 1, 2, \dots) , \\ a_{2n+1} &= \left(-\frac{1}{8}\right)^n a_1^{2n+1} .\end{aligned}\tag{68}$$

The corresponding factor approximants take the same form

$$y_k^*(z) = \frac{8a_1 z}{8 + (a_1 z)^2} \quad (k \geq 3)\tag{69}$$

for orders $k \geq 3$, with the parameter $a_1 = 2\sqrt{2}$ given by the boundary condition (66). Resorting to relations (61) and (63), we obtain

$$\varphi_k^*(x) = \sqrt{2} \operatorname{sech} \left(\sqrt{\frac{2}{\varepsilon}} x \right) \quad (k \geq 3) ,\tag{70}$$

which is the *exact* solution of Eq. (56) describing a bell soliton.

5 Evaluation of approximation accuracy

In the previous sections, we have considered several examples of initial-value and boundary-value problems, for which the self-similar factor approximants result in exact solutions. This fact, as such, that starting with approximations, one can get exact solutions, is highly nontrivial. This is why we have focused our attention on its thorough illustration. But, certainly, the more general situation is when an exact solution cannot be reconstructed, or just does not exist at all. How then could we evaluate the accuracy of our approximate solutions?

A. Solutions defects and errors

Suppose a differential equation, that can be represented in the operator form as

$$E[y(x)] = 0 ,$$

defines a function $y(x)$ of a variable x in the interval $[x_1, x_2]$. The latter can be finite or infinite. Let us find an approximate solution $y_k^*(x)$ of the k -th order. In the theory of differential equations [54,55], one characterizes the accuracy of approximate solutions in two ways, by calculating the solution defects and solution errors.

The *solution defect* of $y_k^*(x)$ is

$$D[y_k^*(x)] \equiv |E[y_k^*(x)]| . \quad (71)$$

This is a local characteristic of an approximate solution $y_k^*(x)$, showing to what extent the considered solution $y_k^*(x)$ does not satisfy the given equation $E[y(x)] = 0$. For the exact solution $y(x)$, the solution defect is, evidently, zero.

Varying x in the whole interval $[x_1, x_2]$, one defines the *maximal solution defect*

$$D[y_k^*] \equiv \sup_x D[y_k^*(x)] , \quad (72)$$

which is the global characteristic of the validity of the approximate solution $y_k^*(x)$, with respect to the given equation $E[y(x)] = 0$, in the whole interval of the variable $x \in [x_1, x_2]$.

The *solution error* of $y_k^*(x)$ is defined as

$$\Delta[y_k^*(x)] \equiv |y_k^*(x) - y(x)| . \quad (73)$$

This local characteristic shows how much the approximate solution $y_k^*(x)$ deviates from the exact solution $y(x)$ at each x .

The *maximal solution error*

$$\Delta[y_k^*] \equiv \sup_x \Delta[y_k^*(x)] \quad (74)$$

gives the global characteristic of the maximal deviation of $y_k^*(x)$ from $y(x)$ in the total interval $[x_1, x_2]$.

Both the solution defects and solution errors characterize the accuracy of approximate solutions. The explicit relation between these characteristics depends on the explicit form of the given differential equation. Also [54,55], one can define the *error-to-defect ratio*

$$\delta[y_k^*] \equiv \frac{\Delta[y_k^*]}{D[y_k^*]} . \quad (75)$$

B. Boundary layer problem

To illustrate the definitions of the previous subsection in the case of the factor approximants, let us consider the boundary-layer problem given by the equation

$$\varepsilon \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - xy = 0 , \quad (76)$$

where the function $y = y(x)$ is defined for x in the interval $0 \leq x \leq 1$, with the boundary conditions

$$y(0) = 0 , \quad y(1) = e . \quad (77)$$

Equation (76) is an example of a nontrivial boundary-layer problem [2,3] for which the conventional matching technique contains not only powers of ε , but also powers of $\sqrt{\varepsilon}$, and $\ln \varepsilon$, which makes the matching rather complicated. Also, perturbation theories in terms of ε do not allow for the exact validity of the boundary conditions (77), but the latter are satisfied only approximately, for small $\varepsilon \ll 1$, making such theories [2,3,12] inapplicable for large ε . Contrary to this, the method of factor approximants, being very simple, gives approximate solutions for arbitrary ε , with high accuracy, and satisfies the boundary conditions exactly for any ε .

It is convenient, first, to redefine the sought function as

$$z \equiv ye^{-x} = z(x) , \quad (78)$$

in order that the latter be varying between zero and one. For function (78), problem (76) transforms to

$$\frac{d^2 z}{dx^2} + \left(2 + \frac{x}{\varepsilon}\right) \frac{dz}{dx} + z = 0 , \quad (79)$$

with the boundary conditions

$$z(0) = 0 , \quad z(1) = 1 . \quad (80)$$

Then we follow the standard procedure, by deriving an expansion of $z(x)$ for asymptotically small $x \rightarrow 0$, and by constructing the factor approximants based on the latter expansion. Finally, using relation (78), we obtain the solutions of Eq. (76) as factor approximants $y_k^*(x)$.

The solution defects (71) for several factor approximants $y_k^*(x)$ are shown in Fig. 1, which demonstrates good uniform convergence of the method. Figure 2 shows the solution errors (73) for the same factor approximants. To better investigate the relation between different accuracy characteristics, we present the maximal solution defects (72), maximal solution error (74), and the error-to-defect ratio (75) in Tables 1 and 2. For small ε , the numerical convergence is slower than for large ε . Therefore, in Table 1 for $\varepsilon = 0.1$, we present the accuracy of higher approximants up to $k = 17$. Numerical convergence for $\varepsilon \geq 1$ is so fast that it is sufficient to consider the approximants up to $k = 7$, as in Table 2. As is seen, the maximal solution errors are much smaller than the maximal solution defects. Hence the latter can serve as an upper bound for the former.

6 Nonlinear radial equations

Now we shall illustrate the power of the method for some physically motivated nonlinear equations in radial variables.

A. Gross-Pitaevskii equation

This is the nonlinear Schrödinger equation applied to different Bose-condensed systems (see Refs. [56–62]). We consider here the variant of this equation describing vortices in Bose systems. Analogous equations describe also vortices in superfluids, superconductors, and Higgs fields. The equation, in dimensionless units, reads as

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r} \frac{d\varphi}{dr} - \frac{\varphi}{r^2} + \varphi - \varphi^3 = 0 , \quad (81)$$

defining a function $\varphi = \varphi(r)$ of the radial variable $r \geq 0$. The boundary conditions are

$$\varphi(0) = 0 , \quad \varphi(\infty) = 1 . \quad (82)$$

The asymptotic behavior of the solution to Eq. (81) at small r is

$$\varphi(r) \simeq \sum_n a_{2n+1} r^{2n+1} \quad (r \rightarrow 0) , \quad (83)$$

where

$$\begin{aligned} a_3 &= -\frac{1}{8} a_1 , & a_5 &= \frac{1 + 8a_1^2}{192} a_1 , \\ a_7 &= -\frac{1 + 80a_1^2}{9216} a_1 , & a_9 &= \frac{1 + 656a_1^2 + 1152a_1^4}{737280} a_1 , \end{aligned}$$

and so on, with a_1 to be defined later from the second of the boundary conditions (82). For convenience, expansion (83) can be rewritten as

$$\varphi(r) \simeq a_1 r \sum_n b_n r^{2n} \quad (r^2 \rightarrow 0) , \quad (84)$$

where $b_n \equiv a_{2n+1}/a_1$. Expression (84) shows that actually, we have an expansion in powers of r^2 . The corresponding factor approximants have the form

$$\varphi_k^*(r) = c_k r \prod_i \left(1 + A_i r^2\right)^{n_i} . \quad (85)$$

The boundary condition

$$\varphi_k^*(0) = 0 \quad (86)$$

for Eq. (85) does not impose additional constraints, being always valid. And the boundary condition

$$\varphi_k^*(\infty) = 1 \quad (87)$$

imposes on Eq. (85) two constraints

$$c_k \prod_i A_i^{n_i} = 1 , \quad 1 + 2 \sum_i n_i = 0 . \quad (88)$$

In labelling the factor approximants, we associate the order k with the number of terms in expansion (84) with respect to r^2 . Then for the second-order approximant φ_2^* , defined by Eq. (85), we have

$$\begin{aligned} c_2 &= 0.518840, & A_1 &= 1, & A_2 &= 0.287401, \\ n_1 &= 0.026243, & n_2 &= -0.526243. \end{aligned}$$

For the third-order approximant $\varphi_3^*(r)$, we get

$$\begin{aligned} c_3 &= 0.585667, & A_1 &= 0.107803, & A_2 &= 0.139245, \\ n_1 &= 1.761220, & n_2 &= -2.261220. \end{aligned}$$

The fourth-order approximant $\varphi_4^*(r)$ is defined by

$$\begin{aligned} c_4 &= 0.585331, & A_1 &= 1, & A_2 &= 0.117323, \\ A_3 &= 0.130445, & n_1 &= -0.000031, & n_2 &= 4.553526, & n_3 &= -5.053495. \end{aligned}$$

For the fifth-order approximant $\varphi_5^*(r)$, we find

$$\begin{aligned} c_5 &= 0.583142, & A_1 &= 0.158576, & A_2 &= 0.045537 + i 0.011910, & A_3 &= A_2^*, \\ n_1 &= -0.994835, & n_2 &= 0.247422 - i 0.429221, & n_3 &= n_2^*. \end{aligned}$$

Similarly, any higher-order approximant can be defined following the standard procedure of Sec. 2. Numerical convergence is quite fast, which is shown in Table 3. Note that the approximate solutions, for the same Eq. (81), found by employing the self-similar root approximants [27], are less accurate than the self-similar factor approximants, considered here. To compare the accuracy of the factor approximants $\varphi_k^*(r)$ and the root approximants $R_k^*(r)$, given in Appendix A, we present in Table 3 both the maximal solution defects $D[\varphi_k^*]$ as well as $D[R_k^*]$. As is seen, the factor approximants are two orders more accurate than the root approximants.

B. Stokes-Oseen equation

A simplified spherically symmetric variant of the Stokes-Oseen equation [1–3] can be written in the form

$$\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} + \varepsilon u \frac{du}{dr} = 0, \quad (89)$$

where $u = u(r)$ and $r \geq 1$. This equation describes the viscous flow past a sphere of unit radius, with ε playing the role of the Reynolds number. The boundary conditions are

$$u(1) = 0, \quad u(\infty) = 1. \quad (90)$$

When considering this equation by means of perturbation theory with respect to the Reynolds number ε , one confronts a very delicate singular boundary-layer problem [2,3], with complicated expansions and matching, involving unexpected orders such as $\varepsilon \ln(1/\varepsilon)$. But in our approach, the problem is easily solvable.

It is convenient, first, to change the variable to

$$x \equiv r - 1 . \quad (91)$$

The function

$$y \equiv u(r(x)) = y(x) \quad (92)$$

obeys the equation

$$\frac{d^2 y}{dx^2} + \frac{2}{1+x} \frac{dy}{dx} + \varepsilon y \frac{dy}{dx} = 0 , \quad (93)$$

with the boundary conditions

$$y(0) = 0 , \quad y(\infty) = 1 . \quad (94)$$

Then, as usual, we construct the factor approximants $y_k^*(x)$ and, using relations (91) and (92), return to the factor approximants $u_k^*(r)$. The maximal defects of the latter are given in Table 4 for different ε . For small Reynolds numbers ε , the numerical convergence is very fast. For $\varepsilon \sim 1$, convergence is a little slower, though again it is easy to reach high accuracy. It is feasible to reach quite good accuracy even for large $\varepsilon \gg 1$, although this requires to construct the factor approximants of higher orders $k \sim 10$.

C. Strongly-singular problem

It is instructive to analyze the equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left(\frac{du}{dr} \right)^2 + \varepsilon u \frac{du}{dr} = 0 , \quad (95)$$

defining the function $u(r)$ for $r \geq 1$, with the same boundary conditions

$$u(1) = 0 , \quad u(\infty) = 1 ,$$

as in Eq. (90). If this equation is treated by perturbation theory [2,3] with respect to ε , it becomes a terribly complicated problem. The matching procedure for that equation is notoriously difficult, because an infinite number of terms, with respect to ε , must be calculated before even the leading order could be explicitly matched [2,3].

In the method of the self-similar factor approximants, the procedure is straightforward, as is described in Sec. 2. We follow the same way as in the previous subsection. A little difference from the previous problem (89) is that for Eq. (95) not all even-order approximants exist, but when they exist, they are close to the odd-order approximants, similarly to the case of Eq. (89). The odd-order approximants exist for all orders we have checked and their maximal defects are presented in Table 5 for different ε , demonstrating good numerical convergence and the possibility of reaching quite good accuracy.

7 Conclusion

We have shown that the method of self-similar factor approximants [35–37,40,50] can serve as a powerful tool for constructing approximate solutions to ordinary differential equations.

These can be the initial-value and boundary-value problems, linear as well as nonlinear differential equations.

A very nontrivial fact is that in some cases factor approximants automatically reconstruct exact solutions even to nonlinear equations, provided such solutions exist. This happens, when the latter solutions can be represented, with the appropriate change of variables, as factor approximants, that is, when the solutions pertain to the class of exactly reproduceable functions [35,50].

The procedure of constructing the factor approximants is very simple, which is described in Sec. 2. This procedure also is uniquely defined. If the asymptotic expansion of a solution has the form of Eq. (1), then all factor approximants enjoy the same structure

$$f_k^*(x) = f_0(x) \prod_i (1 + A_i x)^{n_i} ,$$

whose parameters are defined by the accuracy-through-order procedure.

The quantities A_i and n_i can be functions of all other parameters entering the considered equation. This is why the use of the factor approximants is not limited to the case of small parameters, but is valid for any values of the latter, yielding approximate solutions of high accuracy.

As has been proved in our previous publications [35–37,40,50,63] the self-similar approximants guarantee an essentially higher accuracy than Padé approximants [64]. This fact can be easily understood, if one notices that Padé approximants have the structure which is just a particular variant of the factor approximants. In addition, the factor approximants, contrary to Padé approximants, enjoy a great advantage of being uniquely defined.

Finally, since the quantities A_i and n_i in the expression for the factor approximants can be functions of other parameters, they also can be functions of other variables, entering the equations, and even functions of external fields. This could open the way for extending the suggested technique to solving partial and stochastic differential equations [65]. The latter, however, is the subject for future investigations.

Appendix A

In Sec. 6, subsection 6.1, the factor-approximant solutions $\varphi_k^*(r)$ to the Gross-Pitaevskii equation (81) are compared with the root approximant solutions $R_k^*(r)$. In Table 3, the maximal defects of both types of approximants are given.

The self-similar root approximants [26,27,29,30] are constructed from the asymptotic expansion

$$y(r) \simeq 1 - \frac{1}{2} r^{-2} - \frac{9}{8} r^{-4} - \frac{161}{16} r^{-6}$$

for large $r \rightarrow \infty$, which is an expansion in powers of r^{-2} . The first several root approximants are

$$R_2^*(r) = \frac{r}{2} \left(1 + \frac{1}{4} r^2 \right)^{-1/2},$$

$$R_3^*(r) = \frac{r}{\sqrt{2}} \left(1 + \frac{1}{2} r^2 + \frac{1}{4} r^4 \right)^{-1/4},$$

$$R_4^*(r) = \frac{r}{4^{1/3}} \left(1 + \frac{3}{4} r^2 + \frac{3}{16} r^4 + \frac{1}{16} r^6 \right)^{-1/6},$$

$$R_5^*(r) = \frac{r}{136^{1/8}} \left(1 + r^2 + \frac{9}{68} r^4 + \frac{1}{34} r^6 + \frac{1}{136} r^8 \right)^{-1/8}.$$

The maximal solution defect $D[R_k^*]$ happens at $r \approx 1$ and is shown in Table 3. Though the accuracy of the root approximants is not bad, it is two orders lower than that of the factor approximants.

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Figure Captions

Fig. 1. Solution defects $D[y_k^*(x)]$ of several factor approximants $y_k^*(x)$ with $\varepsilon = 1$, for $k = 4$ (solid line), $k = 5$ (dashed line), $k = 6$ (dotted line), and $k = 7$ (dashed-dotted line), for Eq. (76).

Fig. 2. Solution errors $\Delta[y_k^*(x)]$ of the same factor approximants, as in Fig. 1, for the same orders $k = 4$ (solid line), $k = 5$ (dashed line), $k = 6$ (dotted line), and $k = 7$ (dashed-dotted line), for Eq. (76).

Table Captions

Table 1. Maximal solution defects $D[y_k^*]$, maximal solution errors $\Delta[y_k^*]$, and the error-to-defect ratio $\delta[y_k^*]$ for Eq. (76), with $\varepsilon = 0.1$.

Table 2. Maximal solution defects $D[y_k^*]$, maximal solution errors $\Delta[y_k^*]$, and the error-to-defect ratio $\delta[y_k^*]$ for Eq. (76), with $\varepsilon = 1$ and $\varepsilon = 10$.

Table 3. Maximal solution defects $D[\varphi_k^*]$ of the factor approximants $\varphi_k^*(r)$ and maximal solution defects $D[R_k^*]$ of the root approximants $R_k^*(r)$ for Eq. (81).

Table 4. Maximal solution defects $D[u_k^*]$ of the factor approximants $u_k^*(r)$ for different Reynolds numbers ε , in the case of Eq. (89).

Table 5. Maximal solution defects $D[u_k^*]$ of the factor approximants $u_k^*(r)$ for different ε , in the case of the strongly singular Eq. (95).

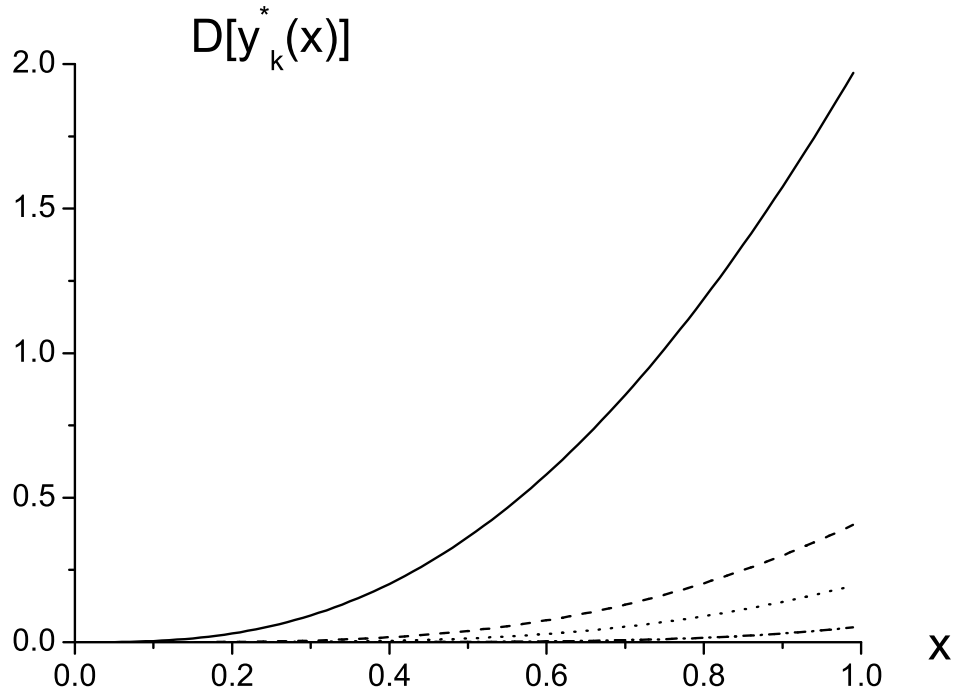


Figure 1: Solution defects $D[y_k^*(x)]$ of several factor approximants $y_k^*(x)$ with $\varepsilon = 1$, for $k = 4$ (solid line), $k = 5$ (dashed line), $k = 6$ (dotted line), and $k = 7$ (dashed-dotted line), for Eq. (76).

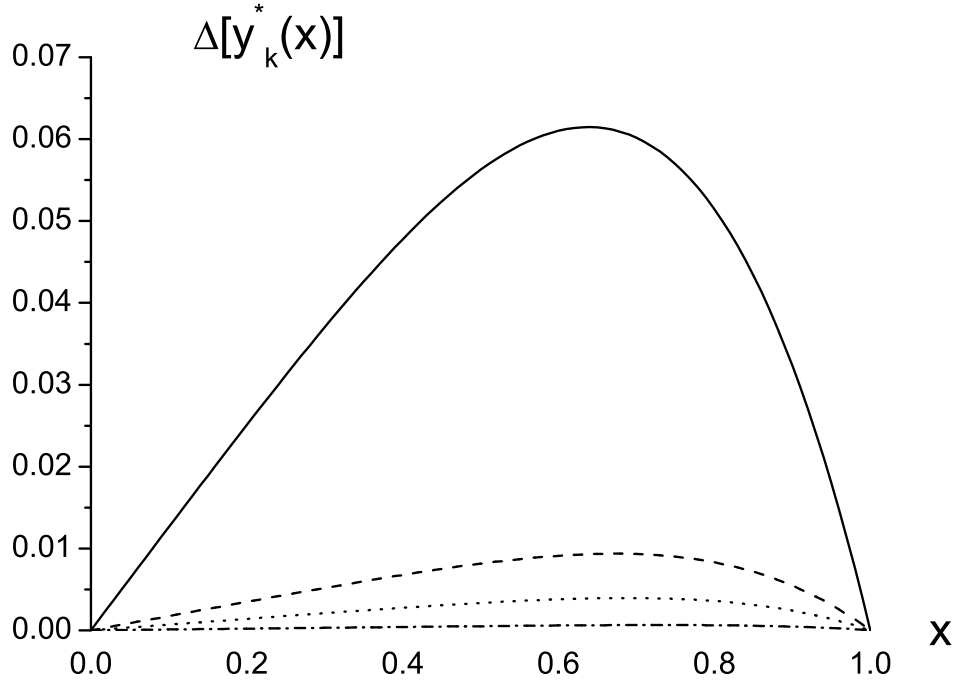


Figure 2: Solution errors $\Delta[y_k^*(x)]$ of the same factor approximants, as in Fig. 1, for the same orders $k = 4$ (solid line), $k = 5$ (dashed line), $k = 6$ (dotted line), and $k = 7$ (dashed-dotted line), for Eq. (76).

k	$D[y_k^*]$	$\Delta[y_k^*]$	$\delta[y_k^*]$
8	1.8	0.14	0.078
9	0.58	0.051	0.088
10	0.20	0.018	0.090
11	0.68	0.038	0.056
12	0.28	0.015	0.054
13	0.090	0.0048	0.053
14	0.024	0.0014	0.058
15	0.057	0.0022	0.039
16	0.032	0.0012	0.038
17	0.0083	0.00028	0.034

Table 1: Maximal solution defects $D[y_k^*]$, maximal solution errors $\Delta[y_k^*]$, and the error-to-defect ratio $\delta[y_k^*]$ for Eq. (76), with $\varepsilon = 0.1$.

		$\varepsilon = 1$			$\varepsilon = 10$	
k	$D[y_k^*]$	$\Delta[y_k^*]$	$\delta[y_k^*]$	$D[y_k^*]$	$\Delta[y_k^*]$	$\delta[y_k^*]$
4	2.0	0.062	0.031	1.58	0.0051	0.0032
5	0.42	0.0094	0.022	0.059	0.00013	0.0022
6	0.20	0.0040	0.020	0.025	0.000046	0.018
7	0.055	0.00064	0.012	0.0032	$3.8 \cdot 10^{-6}$	0.012

Table 2: Maximal solution defects $D[y_k^*]$, maximal solution errors $\Delta[y_k^*]$, and the error-to-defect ratio $\delta[y_k^*]$ for Eq. (76), with $\varepsilon = 1$ and $\varepsilon = 10$.

k	$D[y_k^*]$	$D[R_k^*]$
2	0.12	0.14
3	0.017	0.25
4	0.015	0.10
5	0.0020	0.11
6	0.0018	0.10

Table 3: Maximal solution defects $D[\varphi_k^*]$ of the factor approximants $\varphi_k^*(r)$ and maximal solution defects $D[R_k^*]$ of the root approximants $R_k^*(r)$ for Eq. (81).

k	$\varepsilon = 0.1$	$\varepsilon = 1$	$\varepsilon = 10$
4	0.86	0.094	3.1
5	0.0035	0.015	12
6	0.00068	0.024	40
7	0.00015	0.023	12
8	0.00010	0.00095	0.32
9	0.000056	0.00095	0.15
10	0.000056	0.0015	0.15
11	0.000026	0.0015	0.23
12	0.000024	0.0015	0.011

Table 4: Maximal solution defects $D[u_k^*]$ of the factor approximants $u_k^*(r)$ for different Reynolds numbers ε , in the case of Eq. (89)

k	$\varepsilon = 0.1$	$\varepsilon = 1$	$\varepsilon = 10$
3	0.0035	0.062	1.4
5	0.00056	0.080	0.44
7	0.00060	0.011	0.54
9	0.00035	0.0017	0.32
11	0.00015	0.0017	0.078

Table 5: Maximal solution defects $D[u_k^*]$ of the factor approximants $u_k^*(r)$ for different ε , in the case of the strongly singular Eq. (95).